The criteria of Riesz, Hardy-Littlewood et al. for the Riemann Hypothesis revisited using similar functions

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Abstract. The original criteria of Riesz and of Hardy-Littlewood concerning the truth of the Riemann Hypothesis (RH) are revisited and further investigated in light of the recent formulations and results of Maslanka and of Baez-Duarte concerning a representation of the Riemann Zeta function. Then we introduce a general set of similar functions with the emergence of Poisson-like distributions and we present some numerical experiments which indicate that the RH may barely be true.

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1. Introduction

It is well known that there are many different criteria for the truth of the Riemann Hypothesis (RH). Some of these are not directly related to the important high level computations and developments concerning the non trivial zeros of the Riemann Zeta function. In fact, at the beginning of the century M. Riesz, and later G.H. Hardy and J.E. Littlewood (among other important results in number theory) found a criterion of "classical type" for the truth of the RH. The above criteria are related to some series involving values of the Zeta function outside the critical strip, i.e. at integers arguments of the Zeta function [7, 9], and in a numerical context, very accurate calculations are needed toward a "possible kind of verification" of the RH.

In the literature important remarks have been given by leading mathematicians (see for example, those cited in [3]). We may think that such criteria may have a limited interest since, with them, one should work outside the critical strip. It is, in fact, true that in dealing with the above criteria one needs the use of arguments of the Zeta function outside the critical strip, and problems of interchange of summations are present. As an example, in the above criteria, if one uses the formula established by the authors, one should give a meaning to an integration over the real line, which exists only for finite intervals. In order to obtain finite numerical results which give "satisfactory" values to the functions supposed to be equal to the reciprocal of the Zeta function outside and inside the critical strip, the integration should be carried out using a special sequence of upper limit of integration extending to infinity [6].

But lately, there have been new developments and rigorous results in connection with this kind of problem: first a "regularization" of the representations of the Zeta function (a pioneering work by Maslanka [8]), followed (in particular) by a new rigorous discrete formulation with theorems concerning the above criteria (the works of Baez-Duarte [1, 2, 3]).

In light of these new approaches, we thought that some of the above criteria deserve still more study, at least in the direction of some numerical experiments. Thus, we introduce additional functions containing two parameters, in order to have additional confidence in the numerical results of the experiments.

The content of this work is as follows: in Section 2 we define a general set of functions with two parameters α and β in the spirit of Riesz and of Hardy-Littlewood and then obtain the discrete "representation" of the Zeta function of our set by means of the two parameter Pochammer's polynomials with their coefficients. For the reader the discussion of the conditions are then given in Appendix A and in Appendix B (they follow strictly the ingenious method of Baez-Duarte for the Riesz case $\alpha = \beta = 2$).

In Section 3 we then obtain in some "limit", a Poisson distribution for the coefficients c_k of the Pochammer's polynomials; this is useful in the context of the numerical experiments. These are presented in Section 4 where many various limiting cases are treated. In the case of increasing values of the parameter β, the experiments indicate that the Poisson distribution becomes more and more exact and the discrete function c_k becomes a

constant which can be evaluated. Finally, we present the results of an experiment carried out in the critical strip with different values of $\Re(s)$. As a consequence, we may argue that in the context of the range of validity of the experiments we present, at large and at low values of k, the RH appears to be barely true.

2. The model

We now consider a set of functions with two parameters (α, β) to obtain $\frac{1}{\zeta(s)}$. These are simply an extension of these two cases: the first (with $\alpha = \beta = 2$) introduced and studied by Riesz [9], the second one (where $\alpha = 1$ and $\beta = 2$) by Hardy-Littlewood [7]. Let $\mu(n)$ be the Möbius function of argument n, where

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0, & \text{if } n \text{ contains a square} \end{cases}$$

Let $s_0 = \rho + it$ and s a complex variable. For $\Re(s) > \rho = 1$ one has $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$. The two-parameters family of functions is given by:

$$\varphi(s; \alpha, \beta) := \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} e^{-\frac{x}{n^\beta}} x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx \tag{1}$$

so that expanding the right-hand side in powers of x, we obtain:

$$\varphi(s) = \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! n^{\beta k}} x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx$$

$$= \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \psi(x) x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx$$

where

$$\psi(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \frac{1}{\zeta(\alpha + \beta k)} \tag{2}$$

The function $\psi(x)$ was introduced by Riesz (case $\alpha = \beta = 2$) and by Hardy-Littlewood (case $\alpha = 1, \beta = 2$).

If $\psi(x) \sim \frac{A}{x^{\frac{\alpha-\rho}{\beta}-\epsilon}}$ for some ϵ and for large x, then

$$|\varphi(s)| \leq \left| \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \right| \left| \int_0^\infty \frac{A}{x^{\frac{\alpha-\rho}{\beta}+\frac{\Re(s)-\alpha}{\beta}+1-\epsilon}} dx \right| \leq \left| \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \right| \left| \int_0^\infty \frac{A}{x^{1+\frac{\Re(s)-\rho}{\beta}-\epsilon}} dx \right|$$

would exist and would eventually be given by $\frac{1}{\zeta(s)}$ with $\zeta(s) \neq 0$ if we choose $\Re(s) > \rho + \beta\epsilon$.

Let $\rho = \frac{1}{2}$. For $\alpha = \beta = 2$ we have:

$$\psi(x) = \frac{A}{x^{3/4-\epsilon}}$$

and for $\alpha = 1, \beta = 2$:

$$\psi(x) = \frac{A}{x^{1/4-\epsilon}}$$

On the other hand expanding (1) in a similar way, we have that:

$$\varphi(s) = \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} e^{x(1-\frac{1}{n^\beta})} e^{-x} x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx$$

$$= \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \sum_{k=0}^\infty \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k \int_0^\infty \frac{1}{k!} x^{k-\frac{s-\alpha}{\beta}-1} e^{-x} dx$$

$$= \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \sum_{k=0}^\infty \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k \frac{1}{k!} \Gamma(k - \frac{s-\alpha}{\beta})$$

$$= \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \sum_{k=0}^\infty c_k \prod_{r=1}^k \left(1 - \frac{\frac{s-\alpha}{\beta}+1}{r}\right) \Gamma(-\frac{s-\alpha}{\beta})$$

Thus:

$$\varphi(s) = \sum_{k=0}^\infty c_k P_k(\frac{s-\alpha}{\beta}+1) \tag{3}$$

where $P_k(x) := \prod_{r=1}^k (1 - \frac{x}{r})$ are the Pochhammer polynomials and the functions:

$$c_k := \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k \tag{4}$$

were already studied by Maslanka [8] and Baez-Duarte [2] in the special case $\alpha = \beta = 2$. Let $\Re(s) > \rho + \epsilon$ ($\epsilon > 0$ and $\rho \in [1, \infty)$). From a theorem of Baez-Duarte [2], which says that $|P_k(s)| \leq A \cdot k^{-\Re(s)}$ where A is a constant depending on $|s|$, for large values of k we have that:

$$|\varphi(s)| \leq \sum_{k=0}^\infty |c_k| \, k^{-\left(\frac{\rho+\epsilon-\alpha}{\beta}+1\right)} \tag{5}$$

In Appendix A we show that if $\alpha > 1$ and $\beta > 0$ the following holds unconditionally:

$$q_k \ll \frac{1}{k^{\frac{\alpha-1}{\beta}}}$$

where

$$q_k = \sum_{n=1}^\infty \frac{1}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k$$

Then we obtain:

$$|\varphi(s)| \leq \sum_{k=0}^\infty \frac{1}{k^{\frac{\alpha-1}{\beta}}} \cdot \frac{A}{k^{\frac{\rho+\epsilon-\alpha}{\beta}+1}} \leq \sum_{k=0}^\infty \frac{A}{k^{\frac{\rho+\epsilon-1}{\beta}+1}} \leq A \sum_{k=0}^\infty \frac{1}{k^{1+\frac{\epsilon}{\beta}}} < \infty$$

From this it follows that equation (6) below, which gives $[\zeta(s)]^{-1}$ as:

$$\varphi(s) = \frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k(\frac{s-\alpha}{\beta} + 1) \tag{6}$$

is valid for $\Re(s) > 1$. Now, still from the theorem of Baez-Duarte [2] i.e. that

$$\left| P_k(\frac{s-\alpha}{\beta} + 1) \right| \leq \frac{A}{k^{\frac{\Re(s)-\alpha}{\beta}+1}}$$

and from the hypothesis discussed in Appendix B i.e. supposing the RH to be true for $\Re(s) > \rho + \epsilon$ ($\epsilon > 0$ and $\rho \in [\frac{1}{2}, \infty[$):

$$|c_k| \ll \frac{B}{k^{\frac{1}{\beta}(\alpha-\rho-\epsilon)}}$$

then the above series given by (3) converges uniformely. In fact for $\Re(s) > \rho + \epsilon$ we have:

$$|\varphi(s)| \leq \sum_{k=0}^{\infty} \frac{B}{k^{\frac{1}{\beta}(\alpha-\rho-\epsilon)}} \frac{A}{k^{\frac{\Re(s)-\alpha}{\beta}+1}} \sim \sum_{k=0}^{\infty} \frac{C}{k^{1+\frac{1}{\beta}(\Re(s)-\rho-\epsilon)}}$$

Following Baez-Duarte the series $\varphi(s)$ extends analytically $\frac{1}{\zeta(s)}$ to the half plane $\Re(s) > \rho = \frac{1}{2}$. We have thus obtained for our family of functions with parameters α, β that a necessary and sufficient condition for $\zeta(s) \neq 0$ in the half plane $\Re(s) > \rho$ ($\rho \in [\frac{1}{2}, \infty[$) is given by:

$$|c_k(\alpha,\beta)| \leq \frac{\text{const}}{k^{\frac{1}{\beta}(\alpha-\rho-\epsilon)}} \qquad \forall \epsilon > 0, \forall \alpha > 1, \forall \beta > 0 \tag{7}$$

Remark 1
Instead of using the Möbius function μ in c_k, one may use (for the numerical computations) the formula involving values of the Zeta function:

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(\alpha+\beta j)} \tag{8}$$

Remark 2
From the bound above it follows not only theoretically but also in the context of a numerical analysis that it will be equally difficult to treat the case $\rho \in [\frac{1}{2}, 1]$, for example $\rho = \frac{3}{4}$, as will be the case $\rho = \frac{1}{2} + \epsilon$ with ϵ small (see the last experiment concerning the critical strip where $\rho = \frac{1}{2}$, $\rho = \frac{3}{4}$ and $\rho = 1$. Below in Section 4 we will also treat the case $\alpha = \frac{7}{2}$.

Remark 3
The condition for the truth of the RH using Riesz and Hardy-Littlewood functions $\psi(x)$ is essentially the same as the one using the discrete function c_k with $k \in \mathbb{N}$. In a previous work [6] independent of the present one (which essentially uses the Baez-Duarte idea and theorems) some numerical results were obtained for $\psi(x)$ in the case of the Hardy-Littlewood function ($\alpha = 1, \beta = 2$) by integration in the x-space. The discrete version

using the function c_k of the discrete variable k [2, 8] has advantages in the numerical computations which will be presented below. Before this we present another way to control the function c_k in a numerical context.

3. Poisson like distribution

We still consider the function c_k given by:

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \left(1 - \frac{1}{n^{\beta}} \right)^k$$

Then

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{k \ln(1 - \frac{1}{n^{\beta}})}$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}} e^{k(\ln(1 - \frac{1}{n^{\beta}}) + \frac{1}{n^{\beta}})}$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}} e^{\Delta(k,n,\beta)}$$

Notice that $\Delta < 0$. For β large we set $\Delta = 0$ to obtain the following approximation:

$$c_k \cong \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}}$$

In this approximation we see that c_k of (8) becomes equal to $\psi(x = k)$ of (1). Moreover:

$$c_k \cong \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{k(1 - \frac{1}{n^{\beta}})} e^{-k}$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \sum_{p=0}^{\infty} \frac{k^p}{p!} \left(1 - \frac{1}{n^{\beta}} \right)^p e^{-k}$$

Thus:

$$c_k \cong \sum_{p=0}^{\infty} c_p \frac{k^p}{p!} e^{-k} \tag{9}$$

We are in the presence of a Poisson distribution for the c_p: in this way, in our numerical computations, we may control in a "more satisfactory" way the values of c_k. The approximation for c_k by means of the Poisson distribution for the c_k's we found, will be more satisfactory with increasing values of β and for large values of k. We will also use the approximation given by (9) in which the upper limit of summation will be given by $2k$ instead of ∞, i.e. for large k,

$$c_k \cong \sum_{p=0}^{2k} c_p \frac{k^p}{p!} e^{-k} \tag{10}$$

4. Numerical experiments

4.1. The case $\alpha = \frac{7}{2}$ and $\beta = 4$

This is a case of interest since the behaviour of the c_k at large values of k is expected to be the same as the case $\alpha = \beta = 2$ [2, 8]. In fact from (7) we ask that for $\Re(s) > \frac{1}{2}$:

$$|c_k(7/2, 4)| \leq \frac{C}{k^{\frac{7/2 - 1/2 - \epsilon}{4}}} \sim \frac{k^{\frac{\epsilon}{4}}}{k^{\frac{3}{4}}} \sim |c_k(2, 2)| \tag{11}$$

In the figures 1, 2 and 3 we give the plot respectively of $\log |c_k|$, $\log(|c_k \log k|)$ and $\log(|c_k (\log k)^2|)$ as a function of $\log k$ for k up to 1000 together with the straight line with slope $-\frac{3}{4}$ which is tangent to the curves at some point. The c_k were computed calculating (4) until $n = 10000$.

This experiment indicates that c_k decays more fast than $\frac{C}{(\log k)^2 k^{\frac{3}{4}}}$ as announced by Baez-Duarte in [3] for the case $\alpha = \beta = 2$, i.e. more fast then the bound (11) if the RH is true (see the necessary condition in Appendix B).
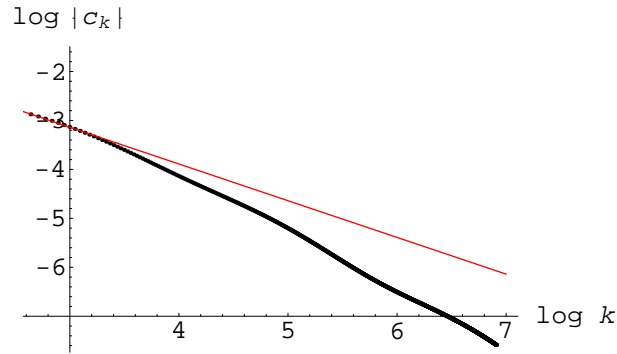

Figure 1. Plot of $\log |c_k| = C - \frac{3}{4} \log k$ together with the straight line of slope $-\frac{3}{4}$.
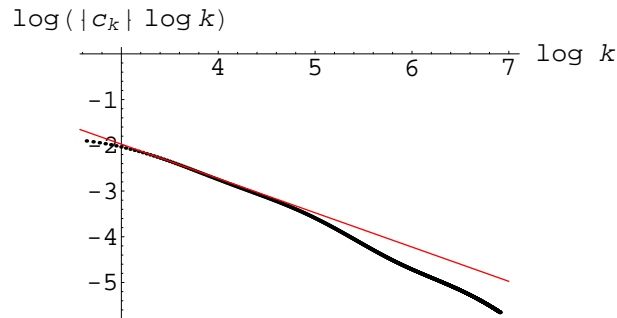

Figure 2. Plot of $\log(|c_k| \log k) = C - \frac{3}{4} \log k$ together with the tangent straight line of slope $-\frac{3}{4}$.
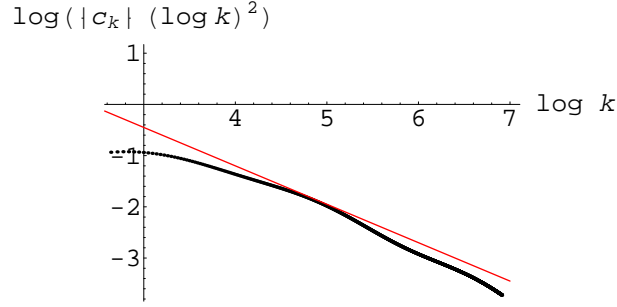
Figure 3. Plot of $\log(|c_k|(\log k)^2) = C - \frac{3}{4}\log k$ together with the tangent straight line of slope $-\frac{3}{4}$.

4.2. The case $\alpha = \frac{2+3\beta}{4}$

If $\alpha = \frac{2+3\beta}{4}$ then all the $c_k(\alpha,\beta)$ are expected to have a decay similar to $k^{-\frac{3}{4}}$ (7). Below (see figure 4) we present the numerical results for β from 1 to 6. This is in agreement with the theory of Appendix B.
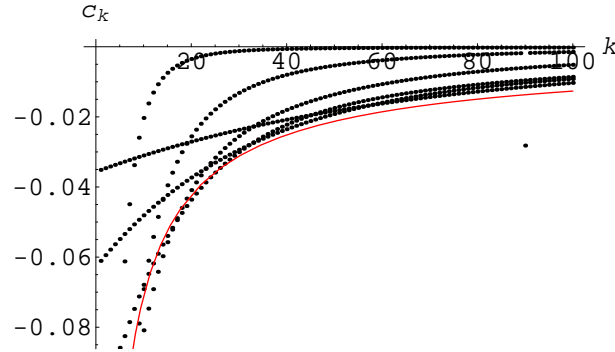

Figure 4. Plot of $c_k(\frac{2+3\beta}{4},\beta)$ for $\beta = 1,2,3,4,5,6$ (dotted lines) vs. the reference curve $f(k) = -0.4k^{-\frac{3}{4}}$.

4.3. The case $\alpha = \frac{7}{2}$ with $\beta \to \infty$

Let $\alpha = \frac{7}{2}$ be fixed, from (8) as β increases we get:

$$\lim_{\beta \to \infty} c_k = \lim_{\beta \to \infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(7/2+\beta j)} = \binom{k}{0}\frac{1}{\zeta(7/2)} - 1 + 1 + \sum_{j=1}^{k} \binom{k}{j}(-1)^j$$

$$= \frac{1}{\zeta(7/2)} - 1 + \sum_{j=0}^{k} \binom{k}{j}(-1)^j$$

Thus:

$$\lim_{\beta \to \infty} c_k = \frac{1}{\zeta(7/2)} - 1 \cong -0.112479 \qquad\qquad \forall k \in \mathbb{N} \qquad\qquad (12)$$
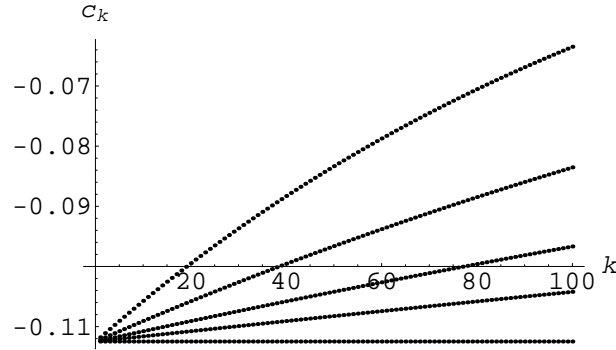
Figure 5. Plot of c_k for $\alpha = 7/2$ and $\beta = 4, 5, 6, 7, 20$ (from top to bottom).

Our numerical experiments convalidate these results. We calculated the first 100 c_k for $\beta = 4, 5, 6, 7, 20$. For $\beta = 20$ we get already a convergence to the theoretical limit (12), see figure 5. So, this infinite β limit obtained by the numerical calculations for low values of k (up to 100) indicates that RH may barely be true (see (7) as $\beta \to \infty$).

4.4. The Poisson distribution

To demonstrate the goodness of the approximation's formula (10) we computed the c_k until $k = 1000$ for the case $\alpha = \frac{7}{2}, \beta = 4$ (using (4)). Then using these already computed c_k we calculated also the first 500 c_k of (10). We ploted these two curves together. In figure 6 we see that from $k \cong 40$ the Poisson approximation is essentially the same as the real function.

4.5. A case with different values of ρ in the critical strip

This experiment is carried out in the critical strip in order to compare the behaviour of c_k in three different cases of ρ. In fact from Appendix B the c_k should all decay at least as:

$$|c_k| < \frac{A}{k^{\frac{3}{4}-\epsilon}} \tag{13}$$

In particular for the cases we treat, i.e. $\alpha = \beta = 2$ (the case of Riesz at $\rho = \frac{1}{2}$), $\alpha = \beta = 3$ at $\rho = \frac{3}{4}$ and $\alpha = \beta = 4$ at $\rho = 1$, the results are presented on figure 7, 8 and 9 respectively. The plots indicate similar behaviour not in disagreement with (13). Further, by looking at these plots, the numerical computations visualize that:

$$|c_k(2,2)| \le |c_k(3,3)| \le |c_k(4,4)|$$

We know that $|c_k(4,4)| \ll k^{-\frac{3}{4}+\epsilon}$ for $\rho \ge 1$ and so $|c_k(3,3)| \ll k^{-\frac{3}{4}+\epsilon}$. This would indicate that there is no zero for $\rho > \frac{3}{4}$. Finally $|c_k(2,2)| \ll k^{-\frac{3}{4}+\epsilon}$ would confirm that there is also no zero for $\rho > \frac{1}{2}$.
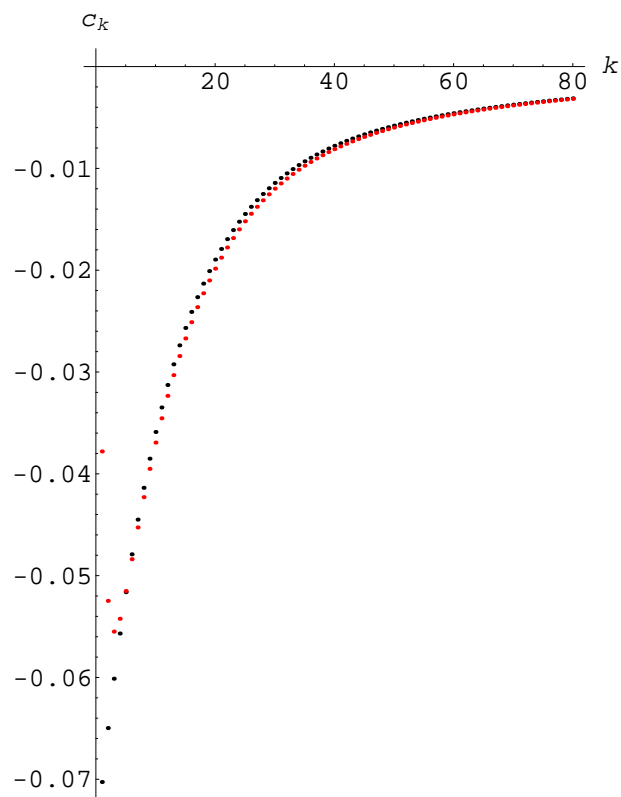
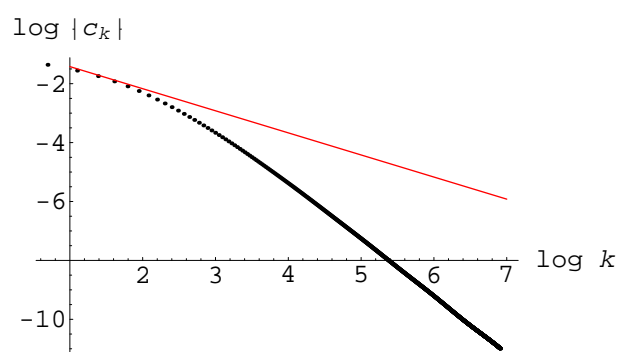**Figure 6.** Plot of c_k for $\alpha = \frac{7}{2}, \beta = 4$ (left) vs. the Poisson approximation (right).

Figure 7. Plot of $\log(|c_k|)$ for $\alpha = \beta = 2$ together with the straight line of slope $-\frac{3}{4}$.
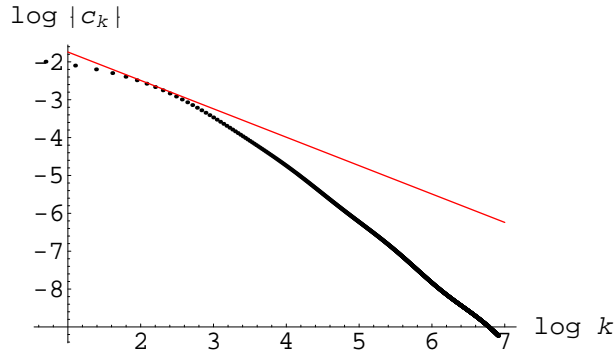
Figure 8. Plot of $\log(|c_k|)$ for $\alpha = \beta = 3$ together with the straight line of slope $-\frac{3}{4}$.
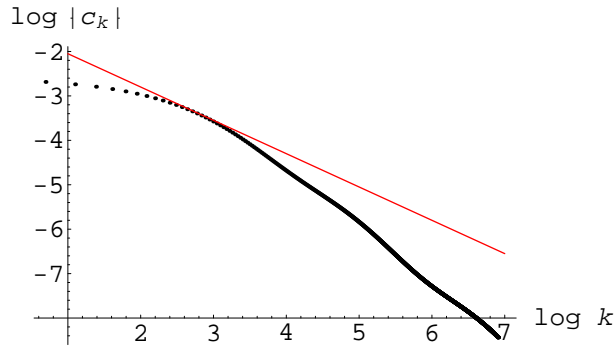

Figure 9. Plot of $\log(|c_k|)$ for $\alpha = \beta = 4$ together with the straight line of slope $-\frac{3}{4}$.

4.6. The case $\alpha = \frac{3}{2}$ and $\beta = 1$

In this case, from the Riesz formulation (1) or from (6), $[\zeta(s)]^{-1}$ is different from infinity if c_k decays at least as k^{-1}. Setting $x = k$ the Riesz function (1) is given by:

$$f(k) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2}} e^{-\frac{k}{n}}$$

Below (figure 10) we present the plot of $f(k)$ together with that of $g(k) = -0.07k^{-1}$ for k up to 600 and for a maximum value of only some hundred for n.

4.7. The case $\alpha = \frac{1}{2}$

In this case ($\alpha < 1$!) we cannot employ the argument of Appendix A, but we have for $\mathfrak{R}(s) - \epsilon \geq \rho = \frac{1}{2}$:

$$|\varphi(s)| < \sum_{k=0}^{\infty} k^{-\left(\frac{\epsilon}{\beta}+1\right)} |c_k(1/2, \beta)| \leq \sum_{k=0}^{\infty} k^{-\left(\frac{\epsilon}{\beta}+1\right)} |c_k(1/2, \infty)|$$

From Subsection 4.3 we know that $|c_k(1/2, \infty)| = |\frac{1}{\zeta(3/2)} - 1| \cong 0.617$, thus for any finite β, $\varphi(s)$ is also finite.
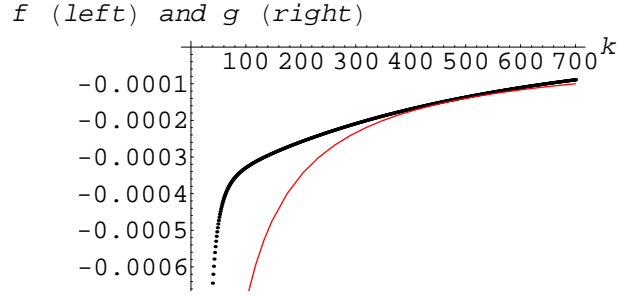
Figure 10. Plot of f and g ($\alpha = \frac{3}{2}, \beta = 1$).

5. Conclusions

In this work we have revisited the original criteria of Riesz and of Hardy and Littlewood for the Riemann Hypothesis in light of recent pioneering works concerning the possible representations of the Riemann Zeta function by means of the Pochammer's polynomials. The discrete representations in the case $\alpha = \beta = 2$ are do to Maslanka and to Baez-Duarte. In order to carry out our numerical experiments related to the criteria, we have first extended the analytical formulation to a more general class of functions containing two parameters α and β; using a theorem of Baez-Duarte we have specified a sufficient and necessary condition for the truth of the RH for our general class of functions i.e. for the decay of the coefficients c_k as a power law of k. Moreover in doing this we have found the emergence of a Poisson-like distribution for the c_k which should be exact in the large β limit. Numerical experiments have been carried out for various cases.

(i) For $\alpha = \frac{7}{2}$ and $\beta = 4$ we have presented intensive calculation using the Möbius function up to $n = 10000$ and for k up to some hundreds. For this case, the power law decay $k^{-\frac{3}{4}}$ is the same as that appearing in the original work of Riesz ($\alpha = \beta = 2$) and also investigated numerically by Baez-Duarte. The experiments confirm the correctness of the power law within the range of the values of n and of k we were able to treat.

(ii) For α and β such that the c_k should all give the power law decay $k^{-\frac{3}{4}}$ at large values of k to ensure the truth of the RH, i.e those where $\alpha = \frac{2+3\beta}{4}$, we have presented some experiments at low values of k and some values of β which confirms this power law decay. All functions c_k assume negative values with plots lying above a fixed curve of equation $y = Ak^{-\frac{3}{4}}$ for some fixed constant A independent of β. In addition for this expected behaviour we have reported the results of intensive computations for the cases $\alpha = \beta = 2$, $\alpha = \beta = 3$ and $\alpha = \beta = 4$.

(iii) Finally we have considered some experiments in the large β limit which indicate that the plots of c_k become more and more flat, well approximated by the mean value of the Poisson-type distribution we have founded. As β becomes large and

large the c_k approaches a constant value, for all k, indicating that in this sense the RH may barely be true.

This work, still accompanied by numerical experiments, may be expanded in the search of other new representations of the Riemann Zeta function, different of the one considered here; moreover there is the aim that the new criteria will be useful in the context of more numerical experiments. The works will be presented in a near future [4, 5].

Appendix A

We follow strictly the lines of calculations of Baez-Duarte [2] to show that the representation (6) for $[\zeta(s)]^{-1}$ is unconditionally valid for $\Re(s) > \rho = 1, \alpha > 1$ and $\beta > 0$. We consider the quantity:

$$q_k = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k$$

Using the MacLaurin series (restricting ourselves to the main contribution), we have that:

$$q_k \cong \int_1^{\infty} \frac{1}{x^\alpha} \left(1 - \frac{1}{x^\beta}\right)^k dx$$

Then with the variable change $y = \frac{1}{x^\beta}$ we obtain:

$$q_k \cong \frac{1}{\beta} \int_0^1 y^{\frac{\alpha-1}{\beta}-1} (1-y)^{k+1-1} dy$$
$$= B(\frac{\alpha-1}{\beta}, k+1)$$

where

$$B(\lambda, \mu) = \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} dx = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}$$

is the Beta function.
Thus for k large, we have

$$q_k \cong \frac{1}{\beta} \Gamma(\frac{\alpha-1}{\beta}) C \frac{k}{k^{\frac{\alpha-1}{\beta}+1}} \cong \frac{1}{k^{\frac{\alpha-1}{\beta}}}$$

Appendix B

Still following Baez-Duarte [2] and here for the family of functions with parameters α and β, we now show the necessity of the condition (7), assuming the RH to be true in the seminfinite strip $\Re(s) > \rho = \frac{1}{2}$.
We set $M(x) = \sum_{n \leq x} \mu(n)$, then we obtain $\forall \epsilon > 0$:

$$M(x) \leq x^{\rho+\epsilon}$$

Integration by parts gives for the main contribution:

$$c_k = \int_1^\infty M(x)\frac{d}{dx}\left(\frac{1}{x^\alpha}\left(1 - \frac{1}{x^\beta}\right)^k\right) dx$$

With the variable change $y = \frac{1}{x}$, using $M(\frac{1}{y}) \ll y^{-\rho-\epsilon}$ for $y \downarrow 0$ (RH) we have:

$$|c_k| < \alpha \int_0^1 y^{\alpha-\rho-\epsilon-1}(1 - y^\beta)^k dy + \beta k \int_0^1 y^{\alpha+\beta-\rho-\epsilon-1}(1 - y^\beta)^{k-1} dy$$

and finally with $y^\beta = z$ we obtain

$$|c_k| < \frac{\alpha}{\beta} \int_0^1 z^{\frac{\alpha-\rho-\epsilon}{\beta}-1}(1 - z)^{k+1-1} dz + k \int_0^1 z^{\frac{\alpha-\rho-\epsilon+\beta}{\beta}-1}(1 - z)^{k-1} dz$$

which for large k is given by:

$$|c_k| < \frac{\alpha}{\beta}\frac{\Gamma(\frac{\alpha-\rho-\epsilon}{\beta})}{k^{\frac{\alpha-\rho-\epsilon}{\beta}}} + \frac{\Gamma(\frac{\alpha-\rho-\epsilon+\beta}{\beta})}{k^{\frac{\alpha-\rho-\epsilon+\beta}{\beta}+1}} < \frac{C}{k^{\frac{\alpha-\rho-\epsilon}{\beta}}}$$

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